Numerical simulation of monuments by the contact dynamics method

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NUMERICAL SIMULATION OF MONUMENTS BY THE CONTACT DYNAMICS METHOD

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SUMMARY

The Non Smooth Contact Dynamics Method (NSCD or CD) is presented in this paper. The purpose of this method is to deal with large collections of rigid or deformable bodies in contact with unilateral constraints and large friction. The method is applied to monuments made of blocks. The relevance of the modelling is discussed. Several examples of buildings statically or dynamically loaded are presented.

1 INTRODUCTION

When buildings made of stone blocks jointed or not with mortar are submitted to dynamical loadings such as earthquakes or quasi-static loadings such as ground level motions, large local stresses are generated in the building causing cracks to appear. Cracks, together with large enough deformations may result in the collapse of the building. On the contrary, cracks may be the manifestation of small displacements between blocks and are not necessarily the premonitory signs of some dangerous situation to happen, displacements between blocks relaxing stresses, and allowing the building to adapt changes in loading. Even sophisticated structures as air planes develop cracks the effect of many of them is relaxing stresses. A reliable structure is thus, not a structure which does not develop cracks, but which generates well distributed not propagating cracks.

Finite elements method applied to a building, considered at first as a single piece continuous body, may provide very interesting results concerning the deformations and the location of largest stresses. Since masonry materials are not much able to bear tensile stresses, a forward step in the analysis would be to cut the meshing where tensile stresses are to appear which usually goes together with a complete remeshing of the damaged area. Since the distribution of stresses is changed when remeshing, a new computation has to be done checking again if tensile stresses are still appearing and if some interpenetration is occurring between contacting elements. Elements have thus to be cut or glued according to the tensile or the interpenetration status. The process is performed until those unilateral constraints are satisfied. Such a "try and error" method is currently used to solve unilateral constraints problems. It may be complicated so as to take into account dry friction. Remeshing is a costly operation and it might happen that the iterative scheme does not converge. It might also give very good results using few iterations. Some special behaviour laws for non tensile materials have also been developed together with Newton-Raphson algorithms to solve quasi-static problems, [8].

The idea of describing a structure as it is, a collection of isolated blocks with unilateral frictional contact between blocks, is thus natural. The overwhelming success of the "Finite Element Method" obliges to name against it "Distinct Elements Method" this modeling method. Distinct elements method goes back to the pioneering work of P. Cundall, [1] and al. used at first to model rocky aggregates, then walls and granular materials, in softwares such as TRUBAL,PFC2D,UDEC,TRIDEC. The Non Smooth Contact Dynamics method or shortly Contact Dynamics (NSCD) has been initiated and developed by M. Jean and J.J. Moreau during the last decade, [5, 6, 7, 2, 4]. M. Jean has developed this method within a Fortran software LMGC. It is a distinct element method but technically quite different from P. Cundall and al. method (PC): roughly, in NSCD, Signorini relation for unilateral conditions and Coulomb law as a dry friction law are adopted together with an implicit algorithm scheme for the dynamical equation, while smooth approximations of these laws are used in PC, together with an explicit scheme. Consequently NSCD uses few large time steps, deals with numerous simultaneous contacts, and needs many iterations at each time step, while PC uses many small time steps, and few iterations at each time step. These methods propagate waves which may be numerically damped in PC to approach quasi-static situations, while NSCD deals with dynamics with small time steps or statics with large time steps and genuine numerical damping generated by implicit methods.

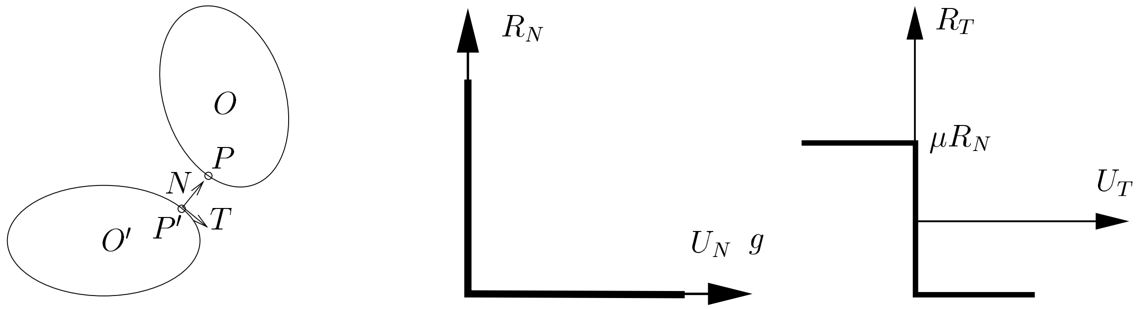
In the next section, the NSCD method is presented. Modelling of buildings is discussed, and some illustrative examples are presented.

2 HINTS ON THE NSCD METHOD

2.1 The frictional contact model, the dynamical equation

More details may be found in [4]. Let O,O', be two neighbouring bodies $P \in O$ and $P' \in O'$, be candidate and antagonist proximal particles. (See figure 1(a)) The vector $\overrightarrow{P'P}$ is a unit normal vector $\vec{N}$ directed from the antagonist object toward the allowed region for the candidate particle. The vector $\vec{N}$ is equipped with two other vectors to form an orthonormal frame, so called local frame. Normal components of vectors in the local frame are denoted with the subscript $_N$ and tangential components orthogonal to $\vec{N}$ with the subscript $_T$. The following mechanical items are used in order to write a frictional contact law:

the components of the relative velocity of P with respect to O' : $U = (U_T, U_N)$,
the components of the reaction force exerted by O' on O : $R = (R_T, R_N)$,
the gap: $g = \overline{P'P}$.



(a) Local frame (b) The Signorini graph (c) Coulomb graph

Figure 1:

Unilateral conditions are:
 impenetrability: $g \geq 0$,
 no attraction is acting between objects: $R_N \geq 0$,
 the reaction force vanishes when the objects are not contacting: $g > 0 \Rightarrow R_N = 0$.
This set of relations may be summarized in the following equivalent complementary relation, the so called *Signorini condition*:

$$g \geq 0 \qquad R_N \geq 0 \qquad gR_N = 0\,. \tag{1}$$

The graph of this relation is displayed figure 1(b). When dynamical situations between rigid bodies are expected, the following relation so-called *velocity Signorini condition* is suitable,

$$\begin{array}{c} \text{at some initial time}\, t_0,\;\; g(t_0) = 0 \;\; ; \\ \text{for all}\; t \in I,\; \text{if}\;\; g(t) = 0 \quad \text{then} \quad U_N^+(t) \geq 0 \quad R_N(t) \geq 0 \quad U_N^+(t)\; R_N(t) = 0\,. \end{array} \tag{2}$$

U^+ is the right relative velocity, the relative velocity after the instant of impact, if any. This relation implies 1 and the satisfaction of the inelastic shock law, (if t is an instant of impact, then $U_N^+(t) = 0$). This shock law seems appropriate for contact between blocks. Cohesion between blocks may be taken into account replacing R_N by $R_N + Coh$ in the above relations, where Coh is some constant which is set to zero as soon as the considered contact is broken.

The basic features of *Coulomb dry friction* are:
 the friction force lies in Coulomb cone: $\|R_T\| \leq \mu R_N$, μ friction coefficient,
 if the sliding velocity is different from zero, the friction force is opposed to the sliding velocity with magnitude μR_N: $U_T^+ \neq 0 \Rightarrow R_T = \mu R_N \dfrac{U_T^+}{\|U_T^+\|}$.
These two conditions may be summarized under the form of a maximum dissipation principle:

$$R_T \in \mathrm{D}(\mu R_N) \quad \forall S \in \mathrm{D}(\mu R_N) \quad (S - R_T)\, U_T^+ \geq 0\,, \tag{3}$$

where, U_T^+ is the right sliding velocity, $\mathrm{D}(R_N)$ is the section of Coulomb cone, the disk with center 0 and radius μR_N. More complicated friction laws may be introduced, for instance making differences between static and dynamic friction coefficient. Coulomb law graph is displayed figure 1(c). Both Signorini and Coulomb graphs are monotonous multi-mapping graphs. The above relations are familiar in the context of Convex Analysis.

The systems under consideration are either collections of rigid blocks or discrete models of deformable blocks. Thus the configuration of the building is given by some generalized variable q, for instance the assembled vector of coordinates of the centers of gravity and rotational components of rigid blocks, or the assembled vector of the node displacements in a finite element description of isolated deformable blocks. Some continuous parts of the building may be described as well using finite elements. The time derivative is denoted $\dot{q}$ (a function of time with bounded variations). For smooth motions the *dynamical equation* writes:

$$M(q)\ddot{q} = F(q, \dot{q}, t) + r \; , \tag{4}$$

where $M(q)$ is the mass matrix; $F(q, \dot{q}, t)$ represents internal forces, elastic forces for instance, external forces, and quadratic acceleration terms; r is the representative of local reaction forces. Since shocks are expected, the derivatives in the above equation are to be understood in the sense of distributions.

Superscripts $^{\alpha}$ $^{\beta}$ are used to denote some candidates to contact $\alpha, \beta \in 1, ..., \chi$. It comes from classical kinematic analysis that the relative velocity U^{α} at some contact α, the derivative of the generalized variable $\dot{q}$, the representative r^{α} of the local reaction force R^{α} for the generalized variable system satisfy the *kinematic relations*,

$$U^{\alpha} = H^{*\alpha}(q)\, \dot{q} \quad , \quad r^{\alpha} = H^{\alpha}(q)\, R^{\alpha} \; . \tag{5}$$

The mappings $H^{\alpha}(q), H^{*\alpha}(q)$, are linear and $H^{*\alpha}(q)$ is the transposed mapping of $H^{\alpha}(q)$. For instance, in the case of a node candidate to contact with a line between two other nodes, the mapping H involve change of variables from the local frame to the general frame and interpolation between velocities of the three nodes. The following relation expressing that the normal component of the relative velocity is the time derivative of the gap function is a key relation as far as discrete forms of unilateral conditions are to be defined,

$$\dot{g}^{\alpha} = U_N^{\alpha} \; . \tag{6}$$

2.2 Discrete forms of the dynamical frictional contact problem

When time discretization is performed, an elementary subinterval $]t_i, t_{i+1}]$ of length h is considered. Integrating both sides of the dynamical equation yields:

$$\left\{ \begin{aligned} M(\dot{q}(t_{i+1}) - \dot{q}(t_i)) &= \int_{t_i}^{t_{i+1}} F(t, q, \dot{q})\, ds + \int_{]t_i, t_{i+1}]} r d\nu, \\ q(t_{i+1}) &= q(t_i) + \int_{t_i}^{t_{i+1}} \dot{q} ds. \end{aligned} \right. \tag{7}$$

The *mean value impulse* denoted $r(i+1)$,

$$r(i+1) = \frac{1}{h} \int_{]t_i, t_{i+1}]} r d\nu \; , \tag{8}$$

emerges as a primary unknown. A numerical scheme is defined by a choice of approximate expressions for the two other integrals in 7. To avoid details needed when embracing the general problem of collections of bodies, the presentation is restricted to the case of small perturbations of rigid or elastic bodies. In this case the mass matrix $M(q)$ is considered as a constant and F takes the form:

$$F(q, \dot{q}, t) = -V\dot{q} - Kq + P(t) \,,$$

where V is the damping matrix and K is the stiffness matrix. Setting $\dot{q}(i)$, $q(i)$, $\dot{q}(i+1)$, $q(i+1)$, respectively approximations of $\dot{q}(t_i)$, $q(t_i)$, $\dot{q}(t_{i+1})$, $q(t_{i+1})$, the Euler implicit scheme writes:

$$\left\{ \begin{array}{c} \dot{q}(i+1) - \dot{q}(i) = w(-hV\dot{q}(i) - hK(q(i) + h\dot{q}(i)) + hP(i+1) + hr(i+1)) \,, \\ q(i+1) - q(i) = h\dot{q}(i+1) \,, \\ \text{where} \\ w = (M + hV + h^2K)^{-1} \quad , \quad P(i+1) = P(t_{i+1}) \,. \end{array} \right. \tag{9}$$

It is assumed that the matrix $M + hV + h^2K$ is invertible, assumption which is satisfied since M is positive definite and V and K are positive.

Writing discrete forms of frictional contact relations needs much care since consistency is to be preserved. According to the above relations 5, the following approximate relative velocities, $U^\alpha(i+1)$, $U^\alpha(i)$, are defined as,

$$U^\alpha(i+1) = H^{*\alpha}(\bar{q})\, \dot{q}(i+1) \,, \quad U^\alpha(i) = H^{*\alpha}(\bar{q})\, \dot{q}(i) \,, \tag{10}$$

where $\bar{q}$ is some auxiliary intermediate value of q, $\bar{q} = q(i)$ being a possible choice. Similarly the following impulses are introduced

$$R^\alpha(i+1) = \frac{1}{h} \int_{]t_i, t_{i+1}]} R^\alpha d\nu \,, \quad r^\alpha(i+1) = H^\alpha(\bar{q})\, R^\alpha(i+1) \,, \tag{11}$$

The construction formula for $q(i+1)$ together with the kinematic relation 6 suggests a predictive formula for the approximate gap,

$$g^\alpha(i+1) = g^\alpha(i) + hU_N^\alpha(i+1) \,. \tag{12}$$

The proposed discrete forms of the Signorini condition 1 and of the velocity Signorini condition 2 are,

$$g(i+1) \geq 0 \qquad R_N(i+1) \geq 0 \qquad g(i+1)\, R_N(i+1) \;=\; 0 \,, \tag{13}$$

if some contact is forecast within the interval $[i, i+1]$ then

$$U_N(i+1) \geq 0 \quad R_N(i+1) \geq 0 \quad U_N(i+1)\, R_N(i+1) = 0 \,. \tag{14}$$

The proposed discrete form for Coulomb law is:

$$\begin{array}{c} R_T^\alpha(i+1) \in \mathrm{D}(\mu R_N^\alpha(i+1)) \\ \forall S \in \mathrm{D}(\mu R_N^\alpha(i+1)) \quad (S - R_T^\alpha(i+1))\, U_T^\alpha(i+1) \geq 0 \,, \end{array} \tag{15}$$

Any of these Signorini conditions, together with Coulomb law may be shortly referred to as

$$\mathrm{SignCoul}\,(i, U^\alpha(i+1), R^\alpha(i+1)) \,. \tag{16}$$

The index i stands for data, $q(i)$, $\dot{q}(i)$, and the gaps known from geometric computations when updating the local frames.

2.3 Solving the basic frictional contact problem

Using the kinematic relations, a linear equation relating relative velocities and mean-values of the impulses may be derived from the discrete form of the linearized equation 9, to be written together with frictional contact equations,

$$U^\alpha(i+1) = U^\alpha_{\text{free}} + \textstyle\sum_\beta W^{\alpha\beta}\, hR^\beta(i+1)\,, \tag{17}$$

$$\text{SignCoul}\,(i, U^\alpha(i+1), R^\alpha(i+1))\,, \tag{18}$$

with,

$$W^{\alpha\beta} = H^{*\alpha}(\bar{q})\; w\; H^\beta(\bar{q}) \quad , \quad U^\alpha_{\text{free}} = H^{*\alpha}(\bar{q})\; v_{\text{free}}\,,$$

$$v_{\text{free}} = \dot{q}(i) + w(-hV\dot{q}(i) - hK(q(i) + h\dot{q}(i)) + hP(i+1))\,.$$

The data are $q(i), \dot{q}(i)$, and the gaps known from geometric computations when updating the local frames. The unknowns are $U^\alpha(i+1), R^\alpha(i+1)$. The algorithm is:

step 1: for a candidate α, assume provisional values of, U^β, R^β, known from the current iteration for $\beta < \alpha$ and known from the previous iteration for $\beta > \alpha$; a straightforward solution U^α, R^α, is found discussing the intersection of graphs of affine mappings (an alternative equivalent form of Signorini Coulomb relations);

step 2: update and proceed to the next candidate;

run over the list of candidates until satisfactory convergence.

This algorithm is similar to a block non-linear Gauss-Seidel algorithm and converges under reasonable mechanical assumptions.

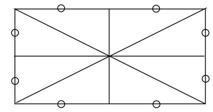
3 MODELING BUILDINGS MADE OF BLOCKS

When studying buildings made of blocks much attention is paid to motions, global deformations and possible appearance of cracks, i.e. openings of joints. The distribution of stresses within and between blocks is more questionable. It is expected, and actually numerically observed, that the results depend very much on the modelling of blocks, on the actual state of stresses within the building, on the way loading is applied and on the kind of algorithm used for the numerical simulation. This particular feature is mainly due to Coulomb frictional contact which may allow an infinity of equilibrium solutions. Physical situations are so. The selected state depends on the history of loading. The choice of an algorithm and the way it is monitored is also in some way part of the history of loading. Nevertheless, numerical simulation shows that some global results may be relevant in spite of the uncertainty mentioned above.

The algorithm used to compute the reaction forces in this application has some particular features. Firstly, as far as the interest is to describe near equilibrium or loosing equilibrium situations, excluding the collapse of the building, small perturbations may be assumed. Secondly, since the number of nodes of each block is very small, and since blocks are independent, the matrix K is a block diagonal matrix, with small band width. The inverse matrix $w = (M + h^2K)^{-1}$ is also a block diagonal matrix with small band width. It results that the elementary operations,

$$v^\alpha = w\, hr^\alpha\,, \quad U^\alpha = H^{*\alpha}v^\alpha\,, \quad r^\alpha = H^\alpha R^\alpha\,,$$

involve a small number of floating point operations. The fully implicit non linear block Gauss-Seidel algorithm presented above is thus reasonably time consuming.



○ candidate for contact

Figure 2: Elementary block 8 T3 and candidates to contact

The global behaviour of the building may be analyzed using the deformation of the building, the crack pattern and the stress field. Introduce the *moment stress tensor* of a rigid or deformable block B,

$$\sigma = \frac{1}{V} \sum_{\alpha \in B} \overrightarrow{OP^{\alpha}} \otimes \overrightarrow{r^{\alpha}} \ ,$$

where $\overrightarrow{r^{\alpha}}$ is the reaction force exerted on the particle candidate to contact P^{α} belonging to the considered block B, O is any point, practically the center of gravity of the block, V is the volume of the block. This tensor is symmetric when the block is at equilibrium. It is practically equal to the mean value of the Cauchy stress tensor on the block. The direction and magnitude of principal stresses are useful results very alike "the thrust line" used in classical idealistic graphic analysis.

A typical model of block used in the presented numerical simulations is described in figure 2. This 2-dimensional block is composed of 8 T3 finite elements. The dots shows midpoints candidates to frictional contact. Some possible choices in the modeling are,

1) the refinement of the block mesh,

2) the use of a consistent mass matrix or a lumped mass matrix or a nodal mass matrix,

3) the use of co-rotational coordinates, i.e. the motion is decomposed into a rigid motion plus a complementary deformation motion, (there exists many such decompositions according to the considered problem and the corresponding mechanical assumptions),

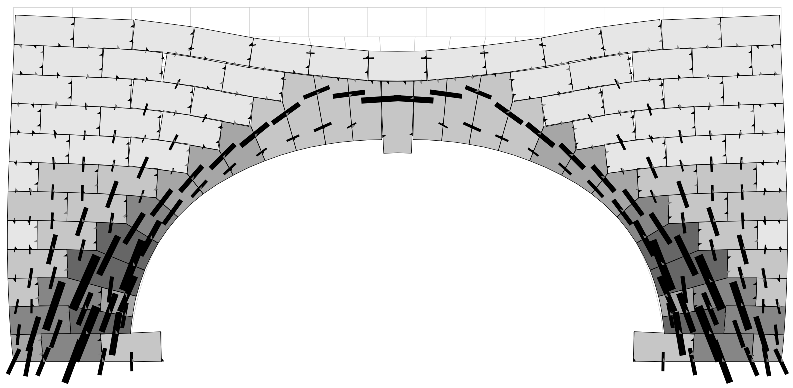
4) the spatial discretization of contact zones, in particular the choice of candidates to contact,

5) the choice of physical modeling, elasticity of blocks, friction coefficients, initial stresses, boundary conditions such as ground level, etc,

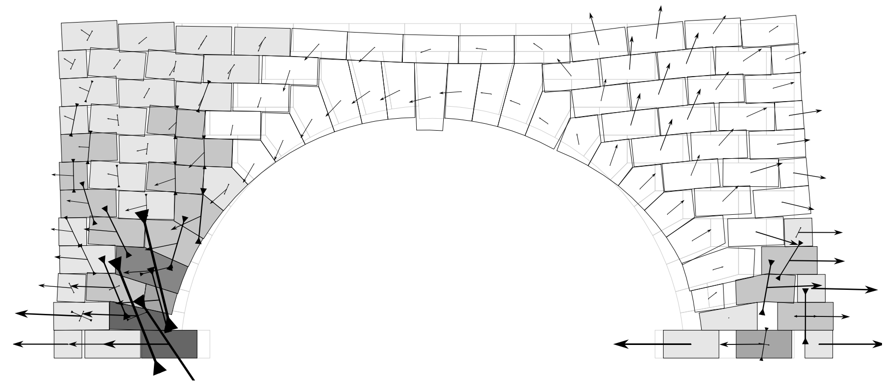
6) the monitoring of the computation, accuracy, error criterion, number of iterations, time step, etc.

The block shapes are not perfectly but approximatively polygonal. The physical data in items 5 are usually badly known. Items 2 or 3 have not much influence in quasi-static situations or as far as low frequencies are concerned. The most sensitive item is the choice of candidates to contact where frictional contact forces are concentrated. A good choice is to concentrate these forces on the supposed center of pressure. Nevertheless due to the meshing, supplementary points may be useful though generating kinematic constraints.

In the examples below (50cm$\times$25cm) elastic blocks, $\rho = 2700$ Kg/m^3, E $= 0.61O^6$ Pa, $\nu = 0.27$, are lying on some rigid foundation. They are roughly meshed as shown in the pictures, using 8 T3 elements for each blocks, see figure 2. The friction coefficient is 0.5 between blocks and blocks and foundation. The gravity load is applied. Pictures show the deformed structure, the distribution of contact forces, and the principal stresses, under various loadings. In those presented simulations item 3 has been used.



(a) Equilibrium under gravity load - Reactions (magnification= 30000)



(b) Velocity and stresses during shear wave excitation (magnification= 200)

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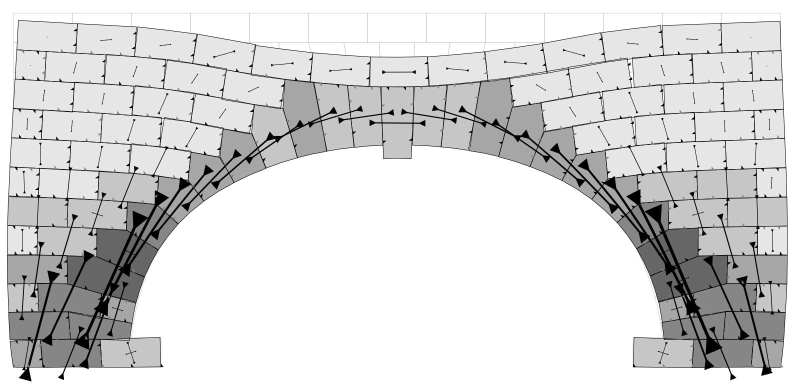
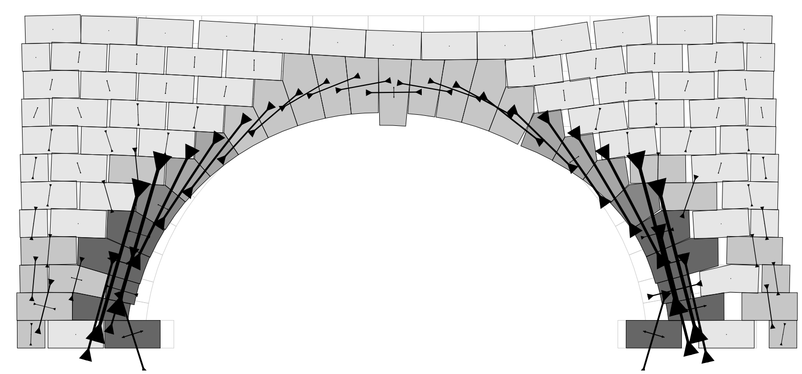
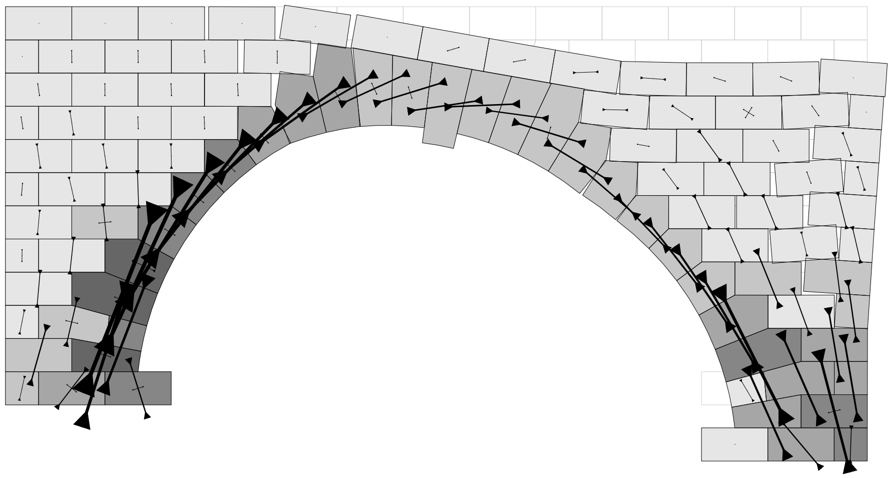


(c) Equilibrium under gravity load - Stresses (magnification= 30000)

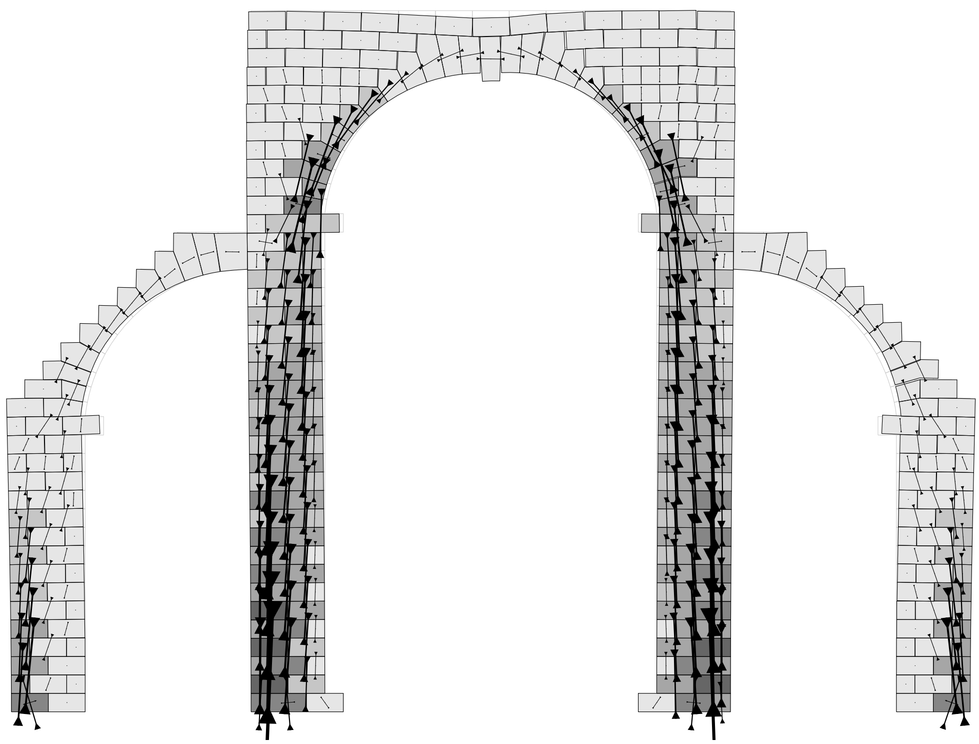


(d) Stresses at equilibrium after shear wave damage (magnification= 200)

Figure 3: Behaviour of an arch under various loads



(a) Stresses after a settlement of ground (-4cm on the right side, magnification = 10)



(b) Stresses in piers and abutments under gravity load (magnification = 300)

Figure 4:

4 CONCLUSION

Other quasi-static or dynamical examples have been treated in the 2 or 3 dimensional cases, [3]. Some 4 T3 elements meshes have also been used. In the 3 dimensional case blocks have been meshed with 8 H8 elements, or more roughly with one single H8 element. There is no inconvenience to use a mixed description, some parts being considered as composed of meshed single piece homogeneous material, other parts where cracks are to appear as composed of distinct blocks. Models of elastic cohesive mortar may be introduced in the frictional contact law. A line of research in collaboration with GAMSAU/MAP, Ecole d'architecture de Marseille-Luminy, France, is to use encoded data from stereophotogrametric pictures to generate finite and discrete elements as to satisfy both mechanics and architecture. Though the numerical simulations produce mechanical likelihood, it is necessary to validate the theoretical model comparing with experimental results. The comparison with experimental results from models of walls made of wood blocks in ESM2, Marseille, France, are encouraging.

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